AP Calculus AB/BC Review

Area of a Region

SOLUTIONS AND SCORING
Question 2

Let \( f \) and \( g \) be the functions defined by \( f(x) = 1 + x + e^{x^2-2x} \) and \( g(x) = x^2 - 6.5x^2 + 6x + 2 \). Let \( R \) and \( S \) be the two regions enclosed by the graphs of \( f \) and \( g \) shown in the figure above.

(a) Find the sum of the areas of regions \( R \) and \( S \).

(b) Region \( S \) is the base of a solid whose cross sections perpendicular to the \( x \)-axis are squares. Find the volume of the solid.

(c) Let \( h \) be the vertical distance between the graphs of \( f \) and \( g \) in region \( S \). Find the rate at which \( h \) changes with respect to \( x \) when \( x = 1.8 \).

\[
\text{Area} = \int_0^2 [g(x) - f(x)] \, dx + \int_A^B [f(x) - g(x)] \, dx
\]

\[
= 0.997427 + 1.006919 = 2.004
\]

\[
\text{Volume} = \int_0^2 [f(x) - g(x)]^2 \, dx = 1.283
\]

\[
h(x) = f(x) - g(x)
\]

\[
h'(x) = f'(x) - g'(x)
\]

\[
h'(1.8) = f'(1.8) - g'(1.8) = -3.812 \text{ (or } -3.811)\]
Let $R$ be the region enclosed by the graph of $f(x) = x^4 - 2.3x^3 + 4$ and the horizontal line $y = 4$, as shown in the figure above.

(a) Find the volume of the solid generated when $R$ is rotated about the horizontal line $y = -2$.

(b) Region $R$ is the base of a solid. For this solid, each cross section perpendicular to the $x$-axis is an isosceles right triangle with a leg in $R$. Find the volume of the solid.

(c) The vertical line $x = k$ divides $R$ into two regions with equal areas. Write, but do not solve, an equation involving integral expressions whose solution gives the value $k$.

(a) $f(x) = 4 \Rightarrow x = 0, 2.3$

Volume = $\pi \int_0^{2.3} \left[ (4 + 2)^2 - (f(x) + 2)^2 \right] dx$

= 98.868 (or 98.867)

(b) Volume = $\int_0^{2.3} \frac{1}{2} (4 - f(x))^2 \, dx$

= 3.574 (or 3.573)

(c) $\int_0^k (4 - f(x)) \, dx = \int_0^{2.3} (4 - f(x)) \, dx$
Let $f(x) = 2x^2 - 6x + 4$ and $g(x) = 4\cos\left(\frac{1}{4} \pi x\right)$. Let $R$ be the region bounded by the graphs of $f$ and $g$, as shown in the figure above.

(a) Find the area of $R$.

(b) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when $R$ is rotated about the horizontal line $y = 4$.

(c) The region $R$ is the base of a solid. For this solid, each cross section perpendicular to the $x$-axis is a square. Write, but do not evaluate, an integral expression that gives the volume of the solid.

(a) Area $= \int_0^2 [g(x) - f(x)] \, dx$

$= \int_0^2 [4\cos\left(\frac{\pi}{4} x\right) - (2x^2 - 6x + 4)] \, dx$

$= \left[4\cdot\frac{4}{\pi}\sin\left(\frac{\pi}{4} x\right) - \left(\frac{2x^3}{3} - 3x^2 + 4x\right)\right]_0^2$

$= \frac{16}{\pi} - \left(\frac{16}{3} - 12 + 8\right) = \frac{16}{\pi} - \frac{4}{3}$

(b) Volume $= \pi \int_0^2 [(4 - f(x))^2 - (4 - g(x))^2] \, dx$

$= \pi \int_0^2 \left[\left(4 - (2x^2 - 6x + 4)\right)^2 - \left(4 - 4\cos\left(\frac{\pi}{4} x\right)\right)^2\right] \, dx$

(c) Volume $= \int_0^2 [g(x) - f(x)]^2 \, dx$

$= \int_0^2 [4\cos\left(\frac{\pi}{4} x\right) - (2x^2 - 6x + 4)]^2 \, dx$
Let $R$ be the region in the first quadrant bounded by the $x$-axis and the graphs of $y = \ln x$ and $y = 5 - x$, as shown in the figure above.

(a) Find the area of $R$.

(b) Region $R$ is the base of a solid. For the solid, each cross section perpendicular to the $x$-axis is a square. Write, but do not evaluate, an expression involving one or more integrals that gives the volume of the solid.

(c) The horizontal line $y = k$ divides $R$ into two regions of equal area. Write, but do not solve, an equation involving one or more integrals whose solution gives the value of $k$.

\begin{align*}
\ln x &= 5 - x \quad \Rightarrow \quad x = 3.69344
\end{align*}

Therefore, the graphs of $y = \ln x$ and $y = 5 - x$ intersect in the first quadrant at the point $(A, B) = (3.69344, 1.30656)$.

\begin{align*}
(a) \quad \text{Area} &= \int_0^A (5 - y - e^y) \, dy \\
&= 2.986 \text{ (or 2.985)} \\
\text{OR} \\
\text{Area} &= \int_1^A \ln x \, dx + \int_A^5 (5 - x) \, dx \\
&= 2.986 \text{ (or 2.985)}
\end{align*}

\begin{align*}
(b) \quad \text{Volume} &= \int_1^A (\ln x)^2 \, dx + \int_A^5 (5 - x)^2 \, dx \\
\text{(c) } \int_0^B (5 - y - e^y) \, dy &= \frac{1}{2} \cdot 2.986 \text{ (or } \frac{1}{2} \cdot 2.985) \\
\end{align*}
AP® CALCULUS AB
2011 SCORING GUIDELINES

Question 3

Let $R$ be the region in the first quadrant enclosed by the graphs of $f(x) = 8x^3$ and $g(x) = \sin(\pi x)$, as shown in the figure above.

(a) Write an equation for the line tangent to the graph of $f$ at $x = \frac{1}{2}$.

(b) Find the area of $R$.

(c) Write, but do not evaluate, an integral expression for the volume of the solid generated when $R$ is rotated about the horizontal line $y = 1$.

\begin{align*}
\text{(a)} & \quad f\left(\frac{1}{2}\right) = 1 \\
& \quad f'(x) = 24x^2, \text{ so } f\left(\frac{1}{2}\right) = 6 \\
\text{An equation for the tangent line is } y &= 1 + 6\left(x - \frac{1}{2}\right). \\
\text{(b)} & \quad \text{Area } = \int_0^{1/2} (g(x) - f(x)) \, dx \\
& \quad = \int_0^{1/2} \left(\sin(\pi x) - 8x^3\right) \, dx \\
& \quad = \left[-\frac{1}{\pi}\cos(\pi x) - 2x^4\right]_{x=0}^{x=1/2} \\
& \quad = -\frac{1}{8} + \frac{1}{\pi} \\
\text{(c)} & \quad \pi \int_0^{1/2} \left((1 - f(x))^2 - (1 - g(x))^2\right) \, dx \\
& \quad = \pi \int_0^{1/2} \left((1 - 8x^3)^2 - (1 - \sin(\pi x))^2\right) \, dx
\end{align*}
Question 3

The functions $f$ and $g$ are given by $f(x) = \sqrt{x}$ and $g(x) = 6 - x$.

Let $R$ be the region bounded by the $x$-axis and the graphs of $f$ and $g$, as shown in the figure above.

(a) Find the area of $R$.

(b) The region $R$ is the base of a solid. For each $y$, where $0 \leq y \leq 2$, the cross section of the solid taken perpendicular to the $y$-axis is a rectangle whose base lies in $R$ and whose height is $2y$. Write, but do not evaluate, an integral expression that gives the volume of the solid.

(c) There is a point $P$ on the graph of $f$ at which the line tangent to the graph of $f$ is perpendicular to the graph of $g$. Find the coordinates of point $P$.

(a) Area = \[ \int_0^4 \sqrt{x} \, dx + \frac{1}{2} \cdot 2 \cdot 2 = \left. \frac{2}{3} x^{3/2} \right|_{x=0}^{x=4} + 2 = \frac{22}{3} \]

(b) $y = \sqrt{x} \Rightarrow x = y^2$

$y = 6 - x \Rightarrow x = 6 - y$

Width = $(6 - y) - y^2$

Volume = \[ \int_0^2 2y(6 - y - y^2) \, dy \]

(c) $g'(x) = -1$

Thus a line perpendicular to the graph of $g$ has slope 1.

\[ f'(x) = \frac{1}{2\sqrt{x}} \]

\[ \frac{1}{2\sqrt{x}} = 1 \Rightarrow x = \frac{1}{4} \]

The point $P$ has coordinates $\left( \frac{1}{4}, \frac{1}{2} \right)$.
Let $R$ be the region in the first quadrant bounded by the graph of $y = 2\sqrt{x}$, the horizontal line $y = 6$, and the $y$-axis, as shown in the figure above.

(a) Find the area of $R$.

(b) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when $R$ is rotated about the horizontal line $y = 7$.

(c) Region $R$ is the base of a solid. For each $y$, where $0 \leq y \leq 6$, the cross section of the solid taken perpendicular to the $y$-axis is a rectangle whose height is $3$ times the length of its base in region $R$. Write, but do not evaluate, an integral expression that gives the volume of the solid.

\[
\text{(a) Area} = \int_0^9 (6 - 2\sqrt{x}) \, dx = \left(6x - \frac{4}{3}x^{3/2}\right)\bigg|_{x=0}^{x=9} = 18
\]

\[
\text{(b) Volume} = \pi \int_0^9 \left((7 - 2\sqrt{x})^2 - (7 - 6)^2\right) \, dx
\]

\[
\text{(c) Solving } y = 2\sqrt{x} \text{ for } x \text{ yields } x = \frac{y^2}{4}.
\]

Each rectangular cross section has area $\left(\frac{3}{4}y^2\right)\left(\frac{y^2}{4}\right) = \frac{3}{16}y^4$.

Volume $= \int_0^6 \frac{3}{16}y^4 \, dy$
Question 1

In the figure above, \( R \) is the shaded region in the first quadrant bounded by the graph of \( y = 4 \ln(3 - x) \), the horizontal line \( y = 6 \), and the vertical line \( x = 2 \).

(a) Find the area of \( R \).
(b) Find the volume of the solid generated when \( R \) is revolved about the horizontal line \( y = 8 \).
(c) The region \( R \) is the base of a solid. For this solid, each cross section perpendicular to the \( x \)-axis is a square. Find the volume of the solid.

\[
\begin{align*}
(a) \ & \int_0^2 (6 - 4 \ln(3 - x)) \, dx = 6.816 \text{ or } 6.817 \\
(b) \ & \pi \int_0^2 \left[ (8 - 4 \ln(3 - x))^2 - (8 - 6)^2 \right] \, dx \\
& = 168.179 \text{ or } 168.180 \\
(c) \ & \int_0^2 (6 - 4 \ln(3 - x))^2 \, dx = 26.266 \text{ or } 26.267
\end{align*}
\]
Let $R$ be the region in the first quadrant enclosed by the graphs of $y = 2x$ and $y = x^2$, as shown in the figure above.

(a) Find the area of $R$.

(b) The region $R$ is the base of a solid. For this solid, at each $x$ the cross section perpendicular to the $x$-axis has area $A(x) = \sin\left(\frac{\pi}{2}x\right)$. Find the volume of the solid.

(c) Another solid has the same base $R$. For this solid, the cross sections perpendicular to the $y$-axis are squares. Write, but do not evaluate, an integral expression for the volume of the solid.

(a) Area $= \int_{0}^{2} (2x - x^2) \, dx$

$= x^2 - \frac{1}{3} x^3 \bigg|_{x=2}^{x=0}$

$= \frac{4}{3}$

(b) Volume $= \int_{0}^{2} \sin\left(\frac{\pi}{2}x\right) \, dx$

$= -\frac{2}{\pi} \cos\left(\frac{\pi}{2}x\right) \bigg|_{x=2}^{x=0}$

$= \frac{4}{\pi}$

(c) Volume $= \int_{0}^{4} \left(\sqrt{y} - \frac{x}{2}\right)^2 \, dy$
Let $R$ be the region bounded by the graphs of $y = \sqrt{x}$ and $y = \frac{x}{2}$, as shown in the figure above.

(a) Find the area of $R$.

(b) The region $R$ is the base of a solid. For this solid, the cross sections perpendicular to the $x$-axis are squares. Find the volume of this solid.

(c) Write, but do not evaluate, an integral expression for the volume of the solid generated when $R$ is rotated about the horizontal line $y = 2$.

(a) Area: $\int_0^4 \left( \sqrt{x} - \frac{x}{2} \right) \, dx = \frac{2}{3} x^{3/2} - \frac{x^2}{4} \bigg|_{x=0}^{x=4} = \frac{4}{3}$

(b) Volume: $\int_0^4 \left( \sqrt{x} - \frac{x}{2} \right)^2 \, dx = \int_0^4 \left( x - \frac{x^{3/2}}{2} + \frac{x^2}{4} \right) \, dx$

$= \frac{x^2}{2} - \frac{2x^{5/2}}{5} + \frac{x^3}{12} \bigg|_{x=0}^{x=4} = \frac{8}{15}$

(c) Volume: $\pi \int_0^4 \left( \left( \frac{2 - x}{2} \right)^2 - (2 - \sqrt{x})^2 \right) \, dx$
AP® CALCULUS AB
2008 SCORING GUIDELINES

Question 1

Let $R$ be the region bounded by the graphs of $y = \sin(\pi x)$ and $y = x^3 - 4x$, as shown in the figure above.

(a) Find the area of $R$.

(b) The horizontal line $y = -2$ splits the region $R$ into two parts. Write, but do not evaluate, an integral expression for the area of the part of $R$ that is below this horizontal line.

(c) The region $R$ is the base of a solid. For this solid, each cross section perpendicular to the $x$-axis is a square. Find the volume of this solid.

(d) The region $R$ models the surface of a small pond. At all points in $R$ at a distance $x$ from the $y$-axis, the depth of the water is given by $h(x) = 3 - x$. Find the volume of water in the pond.

(a) $\sin(\pi x) = x^3 - 4x$ at $x = 0$ and $x = 2$

Area $= \int_0^2 (\sin(\pi x) - (x^3 - 4x)) \, dx = 4$

(b) $x^3 - 4x = -2$ at $r = 0.5391889$ and $s = 1.6751309$

The area of the stated region is $\int_r^s (-2 - (x^3 - 4x)) \, dx$

(c) Volume $= \int_0^2 (\sin(\pi x) - (x^3 - 4x))^2 \, dx = 9.978$

(d) Volume $= \int_0^2 (3 - x)(\sin(\pi x) - (x^3 - 4x)) \, dx = 8.369$ or $8.370$
Let \( R \) be the region in the first quadrant bounded by the graphs of \( y = \sqrt{x} \) and \( y = \frac{x}{3} \).

(a) Find the area of \( R \).

(b) Find the volume of the solid generated when \( R \) is rotated about the vertical line \( x = -1 \).

(c) The region \( R \) is the base of a solid. For this solid, the cross sections perpendicular to the \( y \)-axis are squares. Find the volume of this solid.

The graphs of \( y = \sqrt{x} \) and \( y = \frac{x}{3} \) intersect at the points \((0, 0)\) and \((9, 3)\).

(a) \( \int_0^9 \left( \sqrt{x} - \frac{x}{3} \right) \, dx = 4.5 \)

OR

\( \int_0^3 (3y - y^2) \, dy = 4.5 \)

(b) \( \pi \int_0^3 \left( (3y + 1)^2 - (y^2 + 1)^2 \right) \, dy \)

\( = \frac{207\pi}{5} \approx 130.061 \) or \( 130.062 \)

(c) \( \int_0^3 (3y - y^2)^2 \, dy = 8.1 \)
Let $R$ be the region in the first and second quadrants bounded above by the graph of $y = \frac{20}{1 + x^2}$ and below by the horizontal line $y = 2$.

(a) Find the area of $R$.
(b) Find the volume of the solid generated when $R$ is rotated about the $x$-axis.
(c) The region $R$ is the base of a solid. For this solid, the cross sections perpendicular to the $x$-axis are semicircles. Find the volume of this solid.

\[
\frac{20}{1 + x^2} = 2 \text{ when } x = \pm 3
\]

(a) \[
\text{Area } = \int_{-3}^{3} \left( \frac{20}{1 + x^2} - 2 \right) \, dx = 37.961 \text{ or } 37.962
\]

1: correct limits in an integral in (a), (b), or (c)

2: \[
\begin{align*}
1 & : \text{integrand} \\
1 & : \text{answer}
\end{align*}
\]

(b) \[
\text{Volume } = \pi \int_{-3}^{3} \left( \left( \frac{20}{1 + x^2} \right)^2 - 2^2 \right) \, dx = 1871.190
\]

3: \[
\begin{align*}
2 & : \text{integrand} \\
1 & : \text{answer}
\end{align*}
\]

(c) \[
\begin{align*}
\text{Volume } &= \frac{\pi}{2} \int_{-3}^{3} \left( \frac{1}{2} \left( \frac{20}{1 + x^2} - 2 \right) \right)^2 \, dx \\
&= \frac{\pi}{8} \int_{-3}^{3} \left( \frac{20}{1 + x^2} - 2 \right)^2 \, dx = 174.268
\end{align*}
\]

3: \[
\begin{align*}
2 & : \text{integrand} \\
1 & : \text{answer}
\end{align*}
\]
AP® CALCULUS AB
2007 SCORING GUIDELINES (Form B)

Question 1

Let \( R \) be the region bounded by the graph of \( y = e^{2x-x^2} \) and the horizontal line \( y = 2 \), and let \( S \) be the region bounded by the graph of \( y = e^{2x-x^2} \) and the horizontal lines \( y = 1 \) and \( y = 2 \), as shown above.
(a) Find the area of \( R \).
(b) Find the area of \( S \).
(c) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when \( R \) is rotated about the horizontal line \( y = 1 \).

\[ e^{2x-x^2} = 2 \text{ when } x = 0.446057, 1.553943 \]

Let \( P = 0.446057 \) and \( Q = 1.553943 \)

(a) Area of \( R = \int_{P}^{Q} (e^{2x-x^2} - 2) \, dx = 0.514 \)

(b) \( e^{2x-x^2} = 1 \text{ when } x = 0, 2 \)

\[
\text{Area of } S = \int_{0}^{2} (e^{2x-x^2} - 1) \, dx - \text{Area of } R
\]

\[
= 2.06016 - \text{Area of } R = 1.546
\]

OR

\[
\int_{0}^{P} (e^{2x-x^2} - 1) \, dx + (Q - P) \cdot 1 + \int_{Q}^{2} (e^{2x-x^2} - 1) \, dx
\]

\[
= 0.219064 + 1.107886 + 0.219064 = 1.546
\]

(c) Volume = \( \pi \int_{P}^{Q} \left( \left( e^{2x-x^2} - 1 \right)^2 - (2 - 1)^2 \right) \, dx \)

3 : \{ \text{ integrand } \}

1 : \text{ limits}

1 : answer

3 : \{ \text{ integrand } \}

1 : limits

1 : answer

3 : \{ \text{ constant and limits } \}