AP Calculus AB/BC Review

Value of a Graph

SOLUTIONS AND SCORING
The figure above shows the graph of \( f' \), the derivative of a twice-differentiable function \( f \), on the interval \([-3, 4]\). The graph of \( f' \) has horizontal tangents at \( x = -1 \), \( x = 1 \), and \( x = 3 \). The areas of the regions bounded by the x-axis and the graph of \( f' \) on the intervals \([-2, 1]\) and \([1, 4]\) are 9 and 12, respectively.

(a) Find all \( x \)-coordinates at which \( f' \) has a relative maximum. Give a reason for your answer.

(b) On what open intervals contained in \(-3 < x < 4\) is the graph of \( f \) both concave down and decreasing? Give a reason for your answer.

(c) Find the \( x \)-coordinates of all points of inflection for the graph of \( f \). Give a reason for your answer.

(d) Given that \( f(1) = 3 \), write an expression for \( f(x) \) that involves an integral. Find \( f(4) \) and \( f(-2) \).

(a) \( f'(x) = 0 \) at \( x = -2, x = 1, \) and \( x = 4 \).

\[ f'(x) \] changes from positive to negative at \( x = -2 \).

Therefore, \( f \) has a relative maximum at \( x = -2 \).

(b) The graph of \( f \) is concave down and decreasing on the intervals \(-2 < x < -1\) and \( 1 < x < 3 \) because \( f' \) is decreasing and negative on these intervals.

(c) The graph of \( f \) has a point of inflection at \( x = -1 \) and \( x = 3 \) because \( f' \) changes from decreasing to increasing at these points.

The graph of \( f \) has a point of inflection at \( x = 1 \) because \( f' \) changes from increasing to decreasing at this point.

(d) \[ f(x) = 3 + \int_{1}^{x} f'(t) \, dt \]

\[ f(4) = 3 + \int_{1}^{4} f'(t) \, dt = 3 + (-12) = -9 \]

\[ f(-2) = 3 + \int_{1}^{-2} f'(t) \, dt = 3 - \int_{-2}^{1} f'(t) \, dt \]

\[ = 3 - (-9) = 12 \]
The function \( f \) is defined on the closed interval \([-5, 4]\). The graph of \( f \) consists of three line segments and is shown in the figure above. Let \( g \) be the function defined by \( g(x) = \int_{-3}^{x} f(t) \, dt \).

(a) Find \( g(3) \).

(b) On what open intervals contained in \(-5 < x < 4\) is the graph of \( g \) both increasing and concave down? Give a reason for your answer.

(c) The function \( h \) is defined by \( h(x) = \frac{g(x)}{5x} \). Find \( h'(3) \).

(d) The function \( p \) is defined by \( p(x) = f(x^2 - x) \). Find the slope of the line tangent to the graph of \( p \) at the point where \( x = -1 \).

(a) \( g(3) = \int_{-3}^{3} f(t) \, dt = 6 + 4 - 1 = 9 \)

(b) \( g'(x) = f(x) \)

The graph of \( g \) is increasing and concave down on the intervals \(-5 < x < -3\) and \( 0 < x < 2 \) because \( g' = f \) is positive and decreasing on these intervals.

(c) \( h'(x) = \frac{5xg'(x) - g(x)5}{(5x)^2} = \frac{5xg'(x) - 5g(x)}{25x^2} \)

\[
h'(3) = \frac{(5)(3)g'(3) - 5g(3)}{25 \cdot 3^2} = \frac{15(-2) - 5(9)}{225} = \frac{-75}{225} = -\frac{1}{3} \]

(d) \( p'(x) = f'(x^2 - x)(2x - 1) \)

\( p'(-1) = f'(-1)(2(-1) - 1) = (-2)(-3) = 6 \)
The figure above shows the graph of $f''$, the derivative of a twice-differentiable function $f$, on the closed interval $0 \leq x \leq 8$. The graph of $f''$ has horizontal tangent lines at $x = 1$, $x = 3$, and $x = 5$. The areas of the regions between the graph of $f''$ and the x-axis are labeled in the figure. The function $f$ is defined for all real numbers and satisfies $f(8) = 4$.

(a) Find all values of $x$ on the open interval $0 < x < 8$ for which the function $f$ has a local minimum. Justify your answer.
(b) Determine the absolute minimum value of $f$ on the closed interval $0 \leq x \leq 8$. Justify your answer.
(c) On what open intervals contained in $0 < x < 8$ is the graph of $f$ both concave down and increasing? Explain your reasoning.
(d) The function $g$ is defined by $g(x) = (f'(x))^3$. If $f'(3) = -\frac{5}{2}$, find the slope of the line tangent to the graph of $g$ at $x = 3$.

(a) $x = 6$ is the only critical point at which $f''$ changes sign from negative to positive. Therefore, $f$ has a local minimum at $x = 6$.

(b) From part (a), the absolute minimum occurs either at $x = 6$ or at an endpoint.

\[
f(0) = f(8) + \int_0^8 f''(x) \, dx
= f(8) - \int_0^8 f'(x) \, dx = 4 - 12 = -8
\]

\[
f(6) = f(8) + \int_6^8 f''(x) \, dx
= f(8) - \int_6^8 f'(x) \, dx = 4 - 7 = -3
\]

$f(8) = 4$

The absolute minimum value of $f$ on the closed interval $[0, 8]$ is $-8$.

(c) The graph of $f$ is concave down and increasing on $0 < x < 1$ and $3 < x < 4$, because $f''$ is decreasing and positive on these intervals.

(d) $g'(x) = 3[f'(x)]^2 \cdot f''(x)$

\[
g'(3) = 3[f'(3)]^2 \cdot f''(3) = 3 \left(-\frac{5}{2}\right)^2 \cdot 4 = 75
\]

1 : answer with justification

1 : considers $x = 0$ and $x = 6$

3 : < 1 : answer

1 : justification

2 : 

1 : answer

1 : explanation

3 : 

2 : $g'(x)$

1 : answer

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Let $f$ be the continuous function defined on $[-4, 3]$ whose graph, consisting of three line segments and a semicircle centered at the origin, is given above. Let $g$ be the function given by $g(x) = \int_{1}^{x} f(t) \, dt$.

(a) Find the values of $g(2)$ and $g(-2)$.

(b) For each of $g'(-3)$ and $g''(-3)$, find the value or state that it does not exist.

(c) Find the $x$-coordinate of each point at which the graph of $g$ has a horizontal tangent line. For each of these points, determine whether $g$ has a relative minimum, relative maximum, or neither a minimum nor a maximum at the point. Justify your answers.

(d) For $-4 < x < 3$, find all values of $x$ for which the graph of $g$ has a point of inflection. Explain your reasoning.

| (a) $g(2) = \int_{1}^{2} f(t) \, dt = -\frac{1}{2}(1)(\frac{1}{2}) = -\frac{1}{4}$ | 2 : 1 : $g(2)$  
1 : $g(-2)$ |
| $g(-2) = \int_{-2}^{1} f(t) \, dt = -\int_{-2}^{1} f(t) \, dt$ |  |
| $= -\left(\frac{3}{2} - \frac{\pi}{2}\right) = \frac{\pi}{2} - \frac{3}{2}$ |  |

(b) $g'(x) = f(x)$  
$g'(-3) = f(-3) = 2$  
$g''(x) = f'(x)$  
$g''(-3) = f'(-3) = 1$

(c) The graph of $g$ has a horizontal tangent line where $g'(x) = f(x) = 0$. This occurs at $x = -1$ and $x = 1$.

$g'(x)$ changes sign from positive to negative at $x = -1$. Therefore, $g$ has a relative maximum at $x = -1$.

$g'(x)$ does not change sign at $x = 1$. Therefore, $g$ has neither a relative maximum nor a relative minimum at $x = 1$.

(d) The graph of $g$ has a point of inflection at each of $x = -2$, $x = 0$, and $x = 1$ because $g''(x) = f'(x)$ changes sign at each of these values.
The continuous function $f$ is defined on the interval $-4 \leq x \leq 3$.
The graph of $f$ consists of two quarter circles and one line segment, as shown in the figure above.

Let $g(x) = 2x + \int_0^x f(t) \, dt$.

(a) Find $g(-3)$. Find $g'(x)$ and evaluate $g'(-3)$.

(b) Determine the $x$-coordinate of the point at which $g$ has an absolute maximum on the interval $-4 \leq x \leq 3$.

Justify your answer.

(c) Find all values of $x$ on the interval $-4 < x < 3$ for which the graph of $g$ has a point of inflection. Give a reason for your answer.

(d) Find the average rate of change of $f$ on the interval $-4 \leq x \leq 3$. There is no point $c$, $-4 < c < 3$, for which $f'(c)$ is equal to that average rate of change. Explain why this statement does not contradict the Mean Value Theorem.

(a) $g(-3) = 2(-3) + \int_0^{-3} f(t) \, dt = -6 - \frac{9\pi}{4}$

$g'(x) = 2 + f(x)$

$g'(-3) = 2 + f(-3) = 2$

(b) $g'(x) = 0$ when $f(x) = -2$. This occurs at $x = \frac{5}{2}$.

$g'(x) > 0$ for $-4 < x < \frac{5}{2}$ and $g'(x) < 0$ for $\frac{5}{2} < x < 3$.

Therefore $g$ has an absolute maximum at $x = \frac{5}{2}$.

(c) $g''(x) = f'(x)$ changes sign only at $x = 0$. Thus the graph of $g$ has a point of inflection at $x = 0$.

(d) The average rate of change of $f$ on the interval $-4 \leq x \leq 3$ is

$$\frac{f(3) - f(-4)}{3 - (-4)} = \frac{-2}{7}.$$

To apply the Mean Value Theorem, $f$ must be differentiable at each point in the interval $-4 < x < 3$. However, $f$ is not differentiable at $x = -3$ and $x = 0$.  

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Let \( g \) be the piecewise-linear function defined on \([-2\pi, 4\pi]\) whose graph is given above, and let \( f(x) = g(x) - \cos\left(\frac{x}{2}\right) \).

(a) Find \( \int_{-2\pi}^{4\pi} f(x) \, dx \). Show the computations that lead to your answer.

(b) Find all \( x \)-values in the open interval \((-2\pi, 4\pi)\) for which \( f \) has a critical point.

(c) Let \( h(x) = \int_0^{3x} g(t) \, dt \). Find \( h\left(-\frac{\pi}{3}\right) \).

\[
\int_{-2\pi}^{4\pi} f(x) \, dx = \int_{-2\pi}^{4\pi} \left( g(x) - \cos\left(\frac{x}{2}\right) \right) \, dx \\
= 6\pi^2 - \left[ 2\sin\left(\frac{x}{2}\right) \right]_{x=-2\pi}^{x=4\pi} \\
= 6\pi^2
\]

(b) \( f'(x) = g'(x) + \frac{1}{2}\sin\left(\frac{x}{2}\right) = \begin{cases} 
1 + \frac{1}{2}\sin\left(\frac{x}{2}\right) & \text{for } -2\pi < x < 0 \\
-\frac{1}{2} + \frac{1}{2}\sin\left(\frac{x}{2}\right) & \text{for } 0 < x < 4\pi
\end{cases} \)

\( f''(x) \) does not exist at \( x = 0 \).
For \(-2\pi < x < 0\), \( f''(x) \neq 0 \).
For \(0 < x < 4\pi\), \( f''(x) = 0 \) when \( x = \pi \).

\( f \) has critical points at \( x = 0 \) and \( x = \pi \).

(c) \( h'(x) = g(3x) \cdot 3 \)
\[ h\left(-\frac{\pi}{3}\right) = 3g(-\pi) = 3\pi \]

2: \( \begin{cases} 
1: \text{antiderivative} \\
1: \text{answer}
\end{cases} \)

4: \( \begin{cases} 
1: \frac{d}{dx}\left(\cos\left(\frac{x}{2}\right)\right) \\
1: g'(x) \\
1: x = 0 \\
1: x = \pi
\end{cases} \)

3: \( \begin{cases} 
2: h'(x) \\
1: \text{answer}
\end{cases} \)
The function $g$ is defined and differentiable on the closed interval $[-7, 5]$ and satisfies $g(0) = 5$. The graph of $y = g'(x)$, the derivative of $g$, consists of a semicircle and three line segments, as shown in the figure above.

(a) Find $g(3)$ and $g(-2)$.

(b) Find the $x$-coordinate of each point of inflection of the graph of $y = g(x)$ on the interval $-7 < x < 5$. Explain your reasoning.

(c) The function $h$ is defined by $h(x) = g(x) - \frac{1}{2}x^2$. Find the $x$-coordinate of each critical point of $h$, where $-7 < x < 5$, and classify each critical point as the location of a relative minimum, relative maximum, or neither a minimum nor a maximum. Explain your reasoning.

\[
\begin{align*}
(a) \quad & g(3) = 5 + \int_0^3 g'(x) \, dx = 5 + \frac{\pi \cdot 2^2}{4} + \frac{3}{2} = 13 + \pi \\
& g(-2) = 5 + \int_0^{-2} g'(x) \, dx = 5 - \pi
\end{align*}
\]

\[
\begin{align*}
(b) \quad & \text{The graph of } y = g(x) \text{ has points of inflection at } x = 0, x = 2, \text{ and } x = 3 \text{ because } g' \text{ changes from increasing to decreasing at } x = 0 \text{ and } x = 3, \text{ and } g' \text{ changes from decreasing to increasing at } x = 2.
\end{align*}
\]

\[
\begin{align*}
(c) \quad & h'(x) = g'(x) - x = 0 \Rightarrow g'(x) = x \\
& \text{On the interval } -2 \leq x \leq 2, \quad g'(x) = \sqrt{4 - x^2}.
\end{align*}
\]

\[
\begin{align*}
& \text{On this interval, } g'(x) = x \text{ when } x = \sqrt{2}.
& \text{The only other solution to } g'(x) = x \text{ is } x = 3.
& h'(x) = g'(x) - x > 0 \text{ for } 0 \leq x < \sqrt{2}
& h'(x) = g'(x) - x \leq 0 \text{ for } \sqrt{2} < x \leq 5
\end{align*}
\]

Therefore $h$ has a relative maximum at $x = \sqrt{2}$, and $h$ has neither a minimum nor a maximum at $x = 3$. 

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The derivative of a function \( f \) is defined by
\[
 f'(x) = \begin{cases} 
 g(x) & \text{for } -4 \leq x \leq 0 \\
 5e^{-x/3} - 3 & \text{for } 0 < x \leq 4 
\end{cases}
\]
The graph of the continuous function \( f' \), shown in the figure above, has
\( x \)-intercepts at \( x = -2 \) and \( x = 3\ln\left(\frac{5}{3}\right) \). The graph of \( g \) on \( -4 \leq x \leq 0 \) is a semicircle, and \( f(0) = 5 \).

(a) For \( -4 < x < 4 \), find all values of \( x \) at which the graph of \( f \) has a point of inflection. Justify your answer.

(b) Find \( f(-4) \) and \( f(4) \).

(c) For \( -4 \leq x \leq 4 \), find the value of \( x \) at which \( f \) has an absolute maximum. Justify your answer.

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(a) \( f' \) changes from decreasing to increasing at \( x = -2 \) and from increasing to decreasing at \( x = 0 \). Therefore, the graph of \( f \) has points of inflection at \( x = -2 \) and \( x = 0 \).

\[
f(-4) = 5 + \int_{-4}^{0} g(x) \, dx
= 5 - (8 - 2\pi) = 2\pi - 3
\]

\[
f(4) = 5 + \int_{0}^{4} (5e^{-x/3} - 3) \, dx
= 5 + \left[ -15e^{-x/3} - 3x \right]_{x=0}^{x=4}
= 8 - 15e^{-4/3}
\]

(c) Since \( f'(x) > 0 \) on the intervals \( -4 < x < -2 \) and \( -2 < x < 3\ln\left(\frac{5}{3}\right) \), \( f \) is increasing on the interval \( -4 \leq x \leq 3\ln\left(\frac{5}{3}\right) \).

Since \( f'(x) < 0 \) on the interval \( 3\ln\left(\frac{5}{3}\right) < x < 4 \), \( f \) is decreasing on the interval \( 3\ln\left(\frac{5}{3}\right) \leq x \leq 4 \).

Therefore, \( f \) has an absolute maximum at \( x = 3\ln\left(\frac{5}{3}\right) \).
Let $f$ be a twice-differentiable function defined on the interval $-1.2 < x < 3.2$ with $f(1) = 2$. The graph of $f'$, the derivative of $f$, is shown above. The graph of $f'$ crosses the $x$-axis at $x = -1$ and $x = 3$ and has a horizontal tangent at $x = 2$. Let $g$ be the function given by $g(x) = e^{f(x)}$.

(a) Write an equation for the line tangent to the graph of $g$ at $x = 1$.

(b) For $-1.2 < x < 3.2$, find all values of $x$ at which $g$ has a local maximum. Justify your answer.

(c) The second derivative of $g$ is $g''(x) = e^{f(x)} \left( (f'(x))^2 + f''(x) \right)$. Is $g''(-1)$ positive, negative, or zero? Justify your answer.

(d) Find the average rate of change of $g'$, the derivative of $g$, over the interval $[1, 3]$.

\[
\begin{align*}
\text{(a)} & \quad g(1) = e^{f(1)} = e^2 \\
g'(x) &= e^{f(x)}f'(x), \quad g'(1) = e^{f(1)}f'(1) = -4e^2 \\
The \text{ tangent line is given by } y &= e^2 - 4e^2(x - 1). \\
\end{align*}
\]

\[
\begin{align*}
\text{(b)} & \quad g'(x) = e^{f(x)}f'(x) \\
& \quad e^{f(x)} > 0 \text{ for all } x \\
& \quad \text{So, } g' \text{ changes from positive to negative only when } f' \text{ changes from positive to negative. This occurs at } x = -1 \\
& \quad \text{only. Thus, } g \text{ has a local maximum at } x = -1. \\
\end{align*}
\]

\[
\begin{align*}
\text{(c)} & \quad g''(-1) = e^{f(-1)} \left( (f'(-1))^2 + f''(-1) \right) \\
& \quad e^{f(-1)} > 0 \text{ and } f'(-1) = 0 \\
& \quad \text{Since } f' \text{ is decreasing on a neighborhood of } -1, \\
& \quad f''(-1) < 0. \text{ Therefore, } g''(-1) < 0. \\
\end{align*}
\]

\[
\begin{align*}
\text{(d)} & \quad \frac{g'(3) - g'(1)}{3 - 1} = \frac{e^{f(3)}f'(3) - e^{f(1)}f'(1)}{2} = 2e^2 \\
\end{align*}
\]

2: \{1: \text{difference quotient}, 1: \text{answer}\}

3: \{1: \text{answer}, 1: \text{tangent line equation}\}