Riemann Sums and Definite Integrals

What happens if the intervals aren't even? A big rectangle here, a smaller rectangle there could still work.

- Does it matter, given the amount of rectangles we are using?

- The "long-way" of finding the area under the curve is known as a Riemann Sum.

- Consider the case where the number of rectangles increases and the width of the rectangle decreases.

As the number of rectangles increase, we say that the norm of the partition (or the width of the largest subinterval) decreases.

As $n \to \infty$, $\|\Delta\| \to 0$

This makes another version of our limit-sum definition:

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i) \Delta x_i$$

Definite Integrals

If $f$ is defined on the closed interval $[a, b]$ and the limit

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i) \Delta x_i$$

exists, then $f$ is integrable on $[a, b]$ and the limit is denoted by

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i) \Delta x_i = \int_{a}^{b} f(x) \, dx$$

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The limit is called the **definite integral** of \( f \) from \( a \) to \( b \). The number \( a \) is the **lower limit** of integration, and the number \( b \) is the **upper limit** of integration.

**Continuity Implies Integrability**

If a function \( f \) is continuous on the closed interval \( [a, b] \), then \( f \) is integrable on \( [a, b] \).

**Example**

Evaluate the definite integral \( \int_{-2}^{1} 2x \, dx \)

\[
\Delta x = \frac{b-a}{n} = \frac{3}{n}
\]

\[
c_i = a + i\Delta x = -2 + \frac{3i}{n}
\]

\[
\int_{-2}^{1} 2x \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(c_i)\Delta x
\]

\[
= \lim_{n \to \infty} \sum_{i=1}^{n} 2\left(-2 + \frac{3i}{n}\right)\left(\frac{3}{n}\right)
\]

\[
= \lim_{n \to \infty} \frac{6}{n} \sum_{i=1}^{n} \left(-2 + \frac{3i}{n}\right)
\]

\[
= \lim_{n \to \infty} \frac{6}{n} \left\{-2n + \frac{3}{n}\left(\frac{n(n+1)}{2}\right)\right\}
\]
\[ \lim_{n \to \infty} \left( -12 + 9 + \frac{9}{n} \right) = -3 \]

-This function wasn't non-negative so it muddles the true definition of area!!

**Properties of Definite Integrals**

a) \[ \int_{a}^{b} k \cdot f(x) \, dx = k \int_{a}^{b} f(x) \, dx \]

b) \[ \int_{a}^{b} [f(x) \pm g(x)] \, dx = \int_{a}^{b} f(x) \, dx \pm \int_{a}^{b} g(x) \, dx \]

**Preservation of Inequality**

If \( f \) is integrable and non-negative on \([a, b] \) then

\[ 0 \leq \int_{a}^{b} f(x) \, dx \]

If \( f \) and \( g \) are integrable on the closed interval \([a, b] \) and \( f(x) \leq g(x) \) for every \( x \) in \([a, b] \) then

\[ \int_{a}^{b} f(x) \, dx \leq \int_{a}^{b} g(x) \, dx \]
The Definite Integral as the Area of a Region

If \( f \) is continuous and non-negative on the closed interval \([a, b]\), then the area of the region bounded by the graph of \( f \), the x-axis, and the lines \( x = a \) and \( x = b \) is:

\[
\text{Area} = \int_{a}^{b} f(x) \, dx
\]

Properties of Definite Integrals

If \( f \) is defined at \( x = a \), then we define \( \int_{a}^{a} f(x) \, dx = 0 \)

If \( f \) is integrable on \([a, b]\), then we define \( \int_{a}^{b} f(x) \, dx = -\int_{b}^{a} f(x) \, dx \)

If \( f \) is integrable on 3 closed intervals determined by \( a, b, \) and \( c \), then

\[
\int_{a}^{b} f(x) \, dx = \int_{a}^{c} f(x) \, dx + \int_{c}^{b} f(x) \, dx
\]

Fundamental Theorem of Calculus

Consider the connection between the uses of differentiation and definite integration.

<table>
<thead>
<tr>
<th>Slope</th>
<th>Area</th>
</tr>
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<tbody>
<tr>
<td>( \frac{\Delta y}{\Delta x} )</td>
<td>( \Delta y \Delta x )</td>
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</table>
If a function $f$ is continuous on $[a, b]$ and $F$ is an antiderivative of $f$ on $[a, b]$ then

$$\int_{a}^{b} f(x) \, dx = F(b) - F(a)$$

**Notation**

a) $\int_{a}^{b} f(x) \, dx = F(x) \big|_{a}^{b}$

$$= F(b) - F(a)$$

b) $\int_{a}^{b} f(x) \, dx = \left[ F(x) + C \right]_{a}^{b}$

$$= \left[ F(b) + C \right] - \left[ F(a) + C \right]$$

$$= F(b) - F(a)$$

'So we don't need the constant of integration for definite integrals!'
Example

Evaluate each definite integral.

\[
\int_{1}^{2} (x^2 - 3) \, dx
\]

\[
= \left[ \frac{x^3}{3} - 3x \right]_{1}^{2} = \left( \frac{8}{3} - 6 \right) - \left( \frac{1}{3} - 3 \right) = \frac{2}{3}
\]

\[
\int_{1}^{4} \sqrt[3]{x} \, dx
\]

\[
= 3 \int_{1}^{4} x^{1/2} \, dx = 3 \left[ \frac{x^{3/2}}{3/2} \right]_{1}^{4} = 2(4)^{3/2} - 2(1)^{3/2} = 14
\]

\[
\int_{0}^{\pi/4} \sec^2(x) \, dx
\]

\[
= \tan(x) \bigg|_{0}^{\pi/4} = 1 - 0 = 1
\]

Example

Evaluate \[ \int_{0}^{2} |2x - 1| \, dx \]
\[
|2x - 1| = \begin{cases} 
- (2x - 1) & x < 1/2 \\
2x - 1 & x \geq 1/2 
\end{cases}
\]

Rewrite:

\[
\int_0^2 |2x - 1| \, dx = \int_0^{1/2} -(2x + 1) \, dx + \int_{1/2}^2 (2x - 1) \, dx 
\]
\[
= \left[ -x^2 + x \right]_0^{1/2} + \left[ x^2 - x \right]_{1/2}^2 
\]
\[
= \left( -\frac{1}{4} + \frac{1}{2} \right) - (0 + 0) + (4 - 2) - \left( \frac{1}{2} - \frac{1}{2} \right) = \frac{5}{2}
\]

Example

Find the area of the region bounded by the graph of \( y = 2x^2 - 3x + 2 \), the \( x \)-axis, and the vertical lines \( x = 0 \) and \( x = 2 \).

\[
\text{Area} = \int_0^2 (2x^2 - 3x + 2) \, dx 
\]
\[ = \left[ \frac{2x^3}{3} - \frac{3x^2}{2} + 2x \right]_0 \]

\[ = \left( \frac{16}{3} - 6 + 4 \right) - \left( 0 - 0 + 0 \right) \]

\[ = \frac{10}{3} \]